Calculus Warm Up # 7-1

BC - MC practice:

1) Which series cannot be shown to converge using the limit comparison test with

$$\sum_{n=1}^{\infty} \frac{1}{n^2}\ ?$$

(A) $\sum_{n=1}^{\infty} \frac{4}{3n^2 - n}$

(B) $\sum_{n=1}^{\infty} \frac{15}{\sqrt{n^4} + 5}$

(C) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$

(D) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

2)

$$\int \frac{12}{(x-1)(x-5)}\,dx =$$

(A) $-3\ln|x-1| + 3\ln|x-5| + C$

(B) $-2\ln|x-1| + 2\ln|x-5| + C$

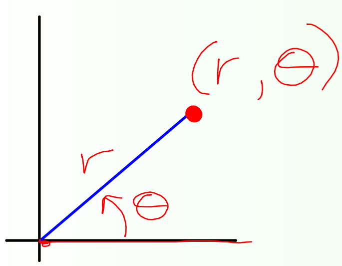
(C) $3\ln|x-1| - 3\ln|x-5| + C$

(D) $12\ln|x-1| + 12\ln|x-5| + C$

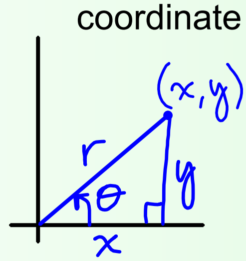
12.4 Polar Coordinates and Equations

(r, θ)

r = directed distance from the pole (origin)
θ = angle of rotation about the pole
(usually from the positive x-axis, r is negative if rotated from the negative x-axis)



Related to the (x, y) coordinate system:



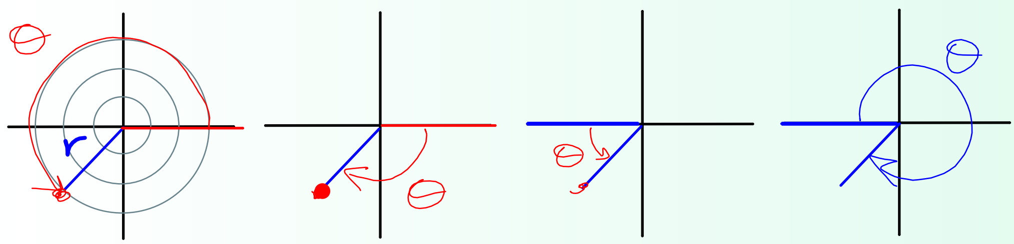
$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

* Other useful relationships: $x^2 + y^2 = r^2$ and $\tan\theta = \dfrac{y}{x}$

Polar coordinates have multiple representations:

$(r, \theta) = (r, \theta \pm 2\pi n)$ Where n = any integer

Ex: $(3, \frac{5\pi}{4}) = (3, -\frac{3\pi}{4}) = (-3, \frac{\pi}{4}) = (-3, -\frac{7\pi}{4})$



$(-r, \theta) = (-r, \theta \pm (2n+1)\pi)$

r is positive when the initial side of θ is on the positive x-axis.
r is negative when the initial side is on the negative x-axis.

Finding slopes and points of tangency for polar curves.

$\frac{dy}{dx}$ (r, θ)

Example polar curve:

$r = 2 - 2\cos\theta$

where $r = f(\theta)$

$f(\theta) = 2 - 2\cos\theta$

$f'(\theta) =$

In General

If r is a differentiable function of θ, $r = f(\theta)$,
then the slope of the tangent is:

$f(\theta) \cdot \sin\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \qquad \begin{array}{l} (y = r\sin\theta) \longrightarrow f(\theta)\cos\theta + f'(\theta)\sin\theta \\ \hline (x = r\cos\theta) \longrightarrow -f(\theta)\sin\theta + f'(\theta)\cos\theta \end{array}$$

$f(\theta)$

You try:

where $r = f(\theta)$

Find $\frac{dy}{dx}$ and slopes of the tangents at (r, θ)

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$r = 2 + 3\sin\theta$ at: $(2, \pi)$ and $(5, \frac{\pi}{2})$

You try:

where $r = f(\theta)$

Find $\frac{dy}{dx}$ and slopes of the tangents at (r, θ)

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$r = 2 + 3\sin\theta$ at: $(2, \pi)$ and $(5, \frac{\pi}{2})$ $\rightarrow$ $\sin\frac{\pi}{2} = 1$

$\cos\frac{\pi}{2} = 0$

slope $= 0$

$f'(\theta) = 3\cos\theta$

$$\frac{dy}{dx} = \frac{(2+3\sin\theta)\cos\theta + 3\cos\theta\sin\theta}{-(2+3\sin\theta)\sin\theta + 3\cos\theta\cos\theta}$$

@ $(2, \pi)$ Slope $= \frac{(2+0)(-1) + 3(-1)(0)}{-(2+0)(0) + 3(-1)(-1)} = -\frac{2}{3}$

$\sin\pi = 0$

$\cos\pi = -1$

Now you can answer MC - A, #23

MC - A, #23

slope of tangent line to $r = 2\theta$ @ $\theta = \frac{\pi}{2}$

$$\frac{dy}{dx} = \frac{2\theta\cos\theta + 2\sin\theta}{-2\theta\sin\theta + 2\cos\theta}$$

$\sin\frac{\pi}{2} = 1$

$\cos\frac{\pi}{2} = 0$

$$\frac{dy}{dx} = -\frac{2}{\pi}$$

10.9 Representing a function with a power series

$f(x) = \frac{1}{1-x}$ looks like: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

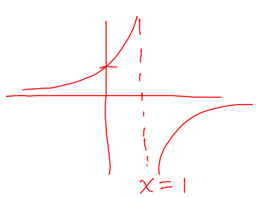
* the sum of a geometric series

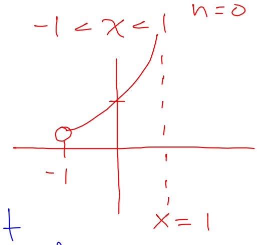
A power series representation of $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \; ; \; |x| < 1$$

Domain of $f(x)$
all $x \neq 1$

Domain for $\sum_{n=0}^{\infty}$
$-1 < x < 1$





good fit
for centered
@ $x = 0$

Centered at $x=-1$; Same function: $f(x)=\frac{1}{1-x}$

$$\frac{1}{1-x}=\frac{1}{1-(x+1)+1}$$

centered @ -1

added to maintain equivalence

Simplified: $\frac{1}{1-x}=\frac{1}{2-(x+1)}$

To write the power series we need to make it look like: $\sum ar^n=\frac{a}{1-r}$

$$\frac{1}{2-(x+1)}=\frac{1}{2\left(1-\left(\frac{x+1}{2}\right)\right)}=\frac{\frac{1}{2}}{1-\left(\frac{x+1}{2}\right)}$$

$a=\frac{1}{2}$

$r=\frac{x+1}{2}$

$$\sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{x+1}{2}\right)^n$$

$$\left|\frac{x+1}{2}\right|<1$$

$$-2<x+1<2$$

$$-3<x<1$$

You try: Find a geometric series centered at x = 0 for f(x)

$f(x) = \frac{4}{x+2}$

Make it look like: $\frac{a}{1 - r}$

$\frac{4}{2(1-(-\frac{x}{2}))} = \frac{2}{1-(-\frac{x}{2})}$

$a = 2$
$r = -\frac{x}{2}$

$f(x) = \sum_{n=0}^{\infty} 2\left(-\frac{x}{2}\right)^n$

$= 2\left[1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} \cdots\right]$

converges when

$\left|-\frac{x}{2}\right| < 1$

$-1 < -\frac{x}{2} < 1$

$-2 < x < 2$

$f(x) = \frac{4}{x+2} = \sum_{n=0}^{\infty} 2\left(-\frac{x}{2}\right)^n$

$= 2\left[1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots\right]$

<u>Using long division to find the series:</u>

$$\begin{array}{r} 2 - x + \frac{x^2}{2} - \frac{x^3}{4} + \cdots \\ 2+x \overline{)\, 4 \qquad\qquad\qquad\qquad} \\ -(4+2x) \qquad\qquad\qquad \\ \hline -2x \qquad\qquad\qquad \\ -(-2x - x^2) \qquad\qquad \\ \hline x^2 \qquad\qquad \\ -\left(x^2 + \frac{x^3}{2}\right) \\ \hline -\frac{x^3}{2} \end{array}$$

Find a geometric series centered at x = 1 for: $f(x) = \frac{1}{x}$

$$\frac{1}{x} = \frac{1}{(x-1)+1} = \frac{1}{1-(1-x)}$$

$$\sum_{n=0}^{\infty} (1-x)^n = 1 + (1-x) + (1-x)^2 + (1-x)^3 + \dots$$

converges when $|1-x| < 1$

$-1 < 1-x < 1$

$0 < x < 2$

HW:

p. 630, # 1 - 13 odd

HW Questions from p. 623

#19, 23, 17, 33

17) $\lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{(n+1)\,5^{n+1}}\cdot\frac{n\,5^n}{(x-5)^n}\right|$

$= \lim_{n\to\infty}\left|\frac{n}{n+1}\right|\cdot\left|\frac{x-5}{5}\right|$

Converges when

$\left|\frac{x-5}{5}\right| < 1$

$-5 < x-5 < 5$

$0 < x < 10$

Endpts

$x=0$ $\sum \frac{(-1)^{n+1}(-5)^n}{n\,5^n}$

$\sum \frac{(-1)^{n+1}(-1)^n\cancel{(5)^n}}{n\,\cancel{5^n}}$

$\sum -\frac{1}{n}$ p-series diverges

$x=10$ $\sum \frac{(-1)^{n+1}(\cancel{5^n})}{n\,\cancel{5^n}}$

$\sum \frac{(-1)^{n+1}}{n}$

Alternating series converges.

Interval of convergence

$0 < x \le 10$

19) $\lim_{n\to\infty}\left|\frac{(x-1)^{n+2}}{n+2}\cdot\frac{n+1}{(x-1)^{n+1}}\right|$

$= \lim_{n\to\infty}\left|\frac{n+1}{n+2}\right|\cdot|x-1|$

Converges when:

$|x-1| < 1$

$0 < x < 2$

endpt $x=2$

$\sum \frac{(-1)^{n+1}}{n+1}$

conditions met for Alternating Series Test

① $\frac{1}{n+2} \le \frac{1}{n+1}$

② $\lim_{n\to\infty}\left(\frac{1}{n+1}\right)=0$

converges

endpts

@ $x=0$

$\sum \frac{(-1)^{n+1}(-1)^{n+1}}{n+1}$

$\sum \frac{1}{n+1}$

Integral Test: pos ✓ cont ✓ decr ✓

0	1	2	3
1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

$\lim_{b\to\infty}\int_1^b \frac{1}{x+1}\,dx$

$\lim_{b\to\infty}\left[\ln(x+1)\right]_1^b$

diverges.

∴ Interval of convergence

$0 < x \le 2$

23) $$\lim_{n\to\infty}\left|\frac{(n+1)(-2x)^n}{(n+2)}\cdot\frac{(n+1)}{n(-2x)^{n-1}}\right|$$

$$\lim_{n\to\infty}\left|\frac{n^2+2n+1}{n^2+2n}\right|\cdot|-2x|$$

converges when:

$$|-2x|<1$$

$$-1<-2x<1$$

$$-\frac{1}{2}<x<\frac{1}{2}$$

Endpoints

$x=-\frac{1}{2}$ n^{th} term test:

$$\sum_{n=1}^{\infty}\frac{n}{n+1} \qquad \lim_{n\to\infty}\frac{n}{n+1}=1\neq 0$$

diverges

$x=\frac{1}{2}$ $\sum\frac{(-1)^{n-1}n}{n+1}$, Alternating Series Test fails $\rightarrow$ $\lim_{n\to\infty}\frac{n}{n+1}=1\neq 0$

diverges

$\therefore$ Interval of Converge: $-\frac{1}{2}<x<\frac{1}{2}$

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33) $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$ series from question #19

a) converges on $0<x\leq 2$

b) $f'(x)=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\cancel{(n+1)}(x-1)^n}{\cancel{(n+1)}}$

$=\sum_{n=0}^{\infty}(-1)^{n+1}(x-1)^n$

interval of convergence
$0<x<2$

endpts:

$x=0$

$\sum_{n=0}^{\infty}(-1)^{n+1}(-1)^n$

$\sum_{n=0}^{\infty}(-1)^{2n+1}$

diverges, $|r|\geq 1$

$x=2$

$\sum(-1)^{n+1}(1)^n$

$\sum(-1)^{n+1}$

diverges

c) $f''(x)=\sum_{n=1}^{\infty}(-1)^{n+1}(n)(x-1)^{n-1}$

$0<x<2$

$\sum(-1)^{n+1}n(-1)^{n-1}$

$\sum(-1)^{2n}n$

$\sum n$

n^{th} term test

$\lim_{n\to\infty}n=\infty\neq 0$

diverges

$\sum(-1)^{n+1}n(1)^{n-1}$

$\sum(-1)^{n+1}n$

Alternating Series Test

① $n+1$ is not $\leq n$

fails

divergce.

d) $\int f(x)dx$

$=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}\cdot\frac{(x-1)^{n+2}}{n+2}$

$0\leq x\leq 2$

$\sum\frac{(-1)^{n+1}(-1)^{n+2}}{(n+1)(n+2)}$

$\sum\frac{(-1)^{2n+3}}{n^2+3n+2}$ ← always odd!

$-\sum\frac{1}{n^2+3n+2}$

$\frac{1}{n^2+3n+2}\leq\frac{1}{n^2}$

larger converges

so $-\sum\frac{1}{n^2+3n+2}$ converges by direct comparison

$\sum\frac{(-1)^{n+1}(1)^{n+2}}{(n+1)(n+2)}$

$\sum\frac{(-1)^{n+1}}{(n+1)(n+2)}$

Alternating Series

① $\frac{1}{(n+2)(n+3)}\leq\frac{1}{(n+1)(n+2)}$

② $\lim_{n\to\infty}\frac{1}{(n+2)(n+3)}=0$

converges by Alt. Series Test